Math 250 Briggs 2/e

4.2 What Derivatives Tell Us (1st of 3)

objectives:

1) Determine intervals where a function is increasing and decreasing
   - interval I can be open (a,b)
     closed $[a,b]$
     half open $[a,b)$ or $(a,b]$
   - include endpoint if $f(a)$ or $f(b)$ is defined

     and

$$\left.\begin{array}{l}\text{increasing if } f'(x)>0\\ \text{decreasing if } f'(x)<0\end{array}\right\} \text{for all points } a<x<b \text{ (interior points)}$$

2) Use the first derivative test to identify local extrema

M250 Intervals of Increasing-Decreasing and 1st Derivative Test for Extrema

A function *f* is <u>increasing</u> on (a,b) if, for any two numbers x_1 and x_2 in (a,b),
If $x_1 < x_2$ then $f(x_1) < f(x_2)$. This means $f'(x) > 0$ for all x in (a,b)

A function *f* is <u>decreasing</u> on (a,b) if, for any two numbers x_1 and x_2 in (a,b),
If $x_1 < x_2$ then $f(x_1) > f(x_2)$. This means $f'(x) < 0$ for all x in (a,b)

A function is <u>constant</u> on (a,b) if, for any two numbers x_1 and x_2 in (a,b),
$f(x_1) = f(x_2)$. This means $f'(x) = 0$ for all x in (a,b)

Since $f'(x)$ gives the slope of the tangent line wherever it is defined on the graph of *f*,

Increasing at $x_1 \Leftrightarrow$ positive slope at $x_1 \Leftrightarrow f'(x_1) > 0$
Decreasing at $x_1 \Leftrightarrow$ negative slope at $x_1 \Leftrightarrow f'(x_1) < 0$
Constant at $x_1 \Leftrightarrow$ zero slope at $x_1 \Leftrightarrow f'(x_1) = 0$

<u>To determine intervals of increasing and decreasing</u>:

1) Find the critical values, where f' is zero or undefined.
2) Graph the critical values in numerical order on a number line.
3) Label the number line f'
4) Determine the sign of the derivative at a test point for each interval between critical values.
5) Write open intervals, identifying those with positive f' at the test point as increasing, and those with negative f' at the test point as decreasing.

<u>CAUTION</u>: When testing values, be sure to test using the <u>derivative</u> f', not the original function *f*.

<u>NOTE</u>: Sometimes it is easier to look at the graph of f'. If the graph of f' is above the x-axis, its value is positive. If the graph of f' is below the x-axis, its value is negative.

<u>To determine if a critical value is an extremum using the first derivative test</u>:

Do steps 1-4 as in determining intervals of increasing or decreasing, above.

5) For each critical value, observe if the sign of f' changed from the left to the right side of that value.
    - If the sign is the same (+ on both sides, or – on both sides), the critical value in not an extremum.
    - If the sign changes from + to –, the slope goes from positive (upward) to negative (downward), making the critical value a relative maximum.
    - If the sign changes from – to +, the slope goes from negative (downward) to positive (upward), making the critical value a relative minimum.
    - Find the y-coordinate for each critical value. (If *f* is undefined, even if there is a sign change, there is <u>not</u> an extremum.)

Test for Increase and Decrease

If f is continuous on an interval I
f is differentiable on all interior points of I (open interval)
$f'(x) > 0$ for all interior points of I
then f is increasing on I.

Ditto: $f'(x) < 0 \Rightarrow f$ is decreasing

Technically, this theorem could result in closed interval answers, but in practice, this author uses only open intervals.

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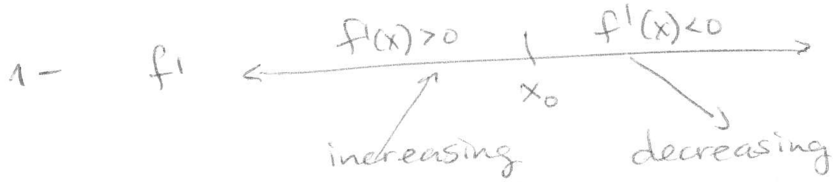
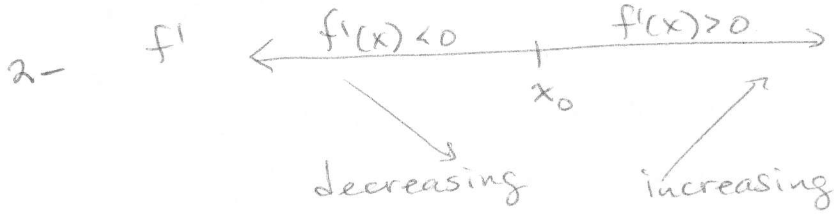
To find relative extrema, we need to test each critical value to confirm that it is relatively larger (or smaller) than nearby values. There are two tests

First Derivative Test:

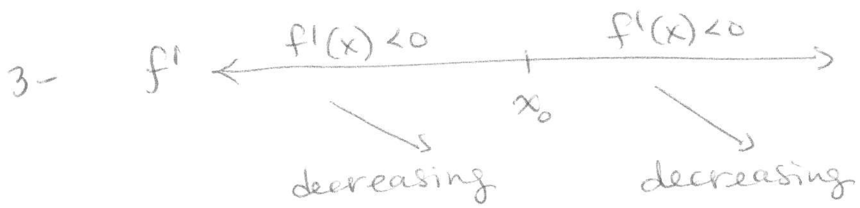
Consider $f'(x)$, the slope of the tangent, to the left and to the right of the critical value x_0.



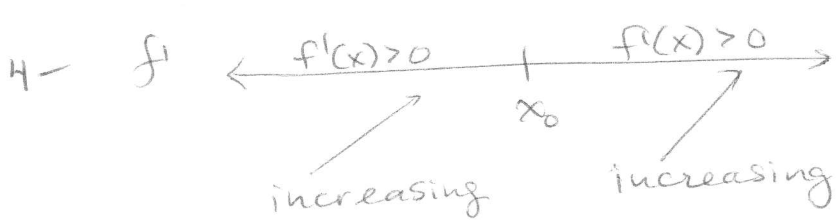
relative maximum $f(x_0)$ at x_0



relative minimum $f(x_0)$ at x_0



$f(x_0)$ is not a relative extremum



$f(x_0)$ is not a relative extremum.

A graph of this type is called a <u>sign chart</u>. Sign charts can be made for f, f' or f'', so always label yours to indicate which function was used.

A point x_0 where $f'(x_0)=0$ is called a <u>stationary point</u> because at that instant, the rate of change is 0.

<u>Advantage of 1st derivative test</u>: It is always conclusive

<u>Disadvantage of 1st derivative test</u>: It may take more work. (not always

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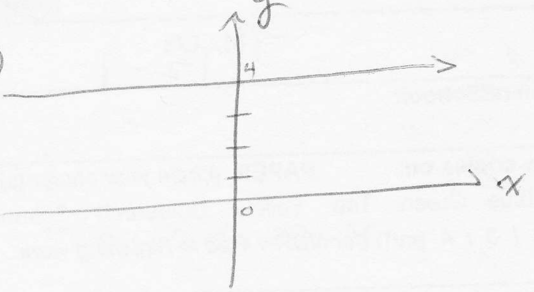
A graph can also be neither increasing nor decreasing:

✓ Ex. ①

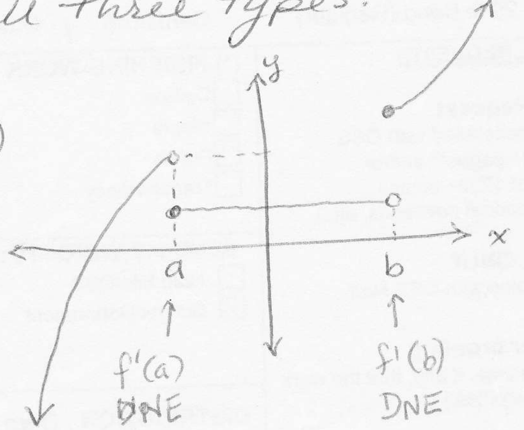


$f(x)=4$ neither increases nor decreases.

f is constant on $(-\infty,\infty)$

A function may have intervals of all three types:

X Ex. ②



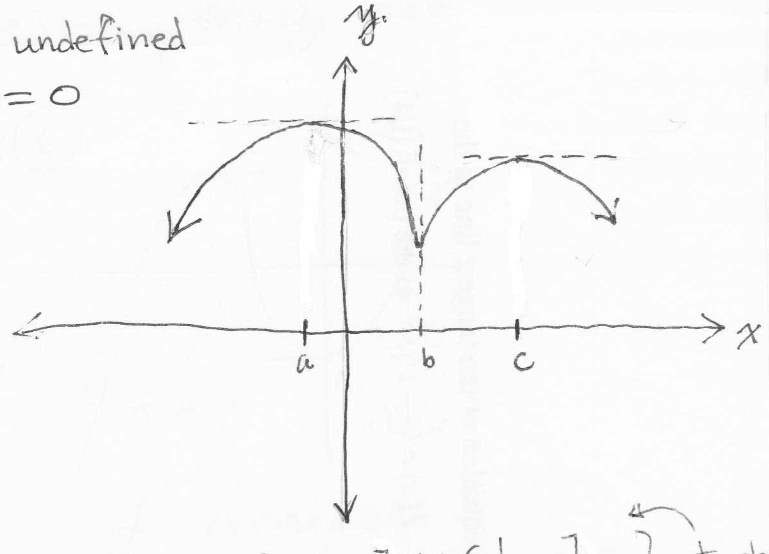
f increases $(-\infty, a)$

f is constant $[a, b)$ ← technically! but $\boxed{(a,b)}$

f decreases (b, ∞)

$f'(a)=0$
$f'(b)$ undefined
$f'(c)=0$

✓ Ex ③



There may be several intervals that are not contiguous.

increases $(-\infty,a] \cup (b, c]$
decreases $[a,b) \cup [c,\infty)$
} technically!

prob. $(-\infty,a)\cup(b,c)$
$(a,b)\cup(c,\infty)$

Notice: A function changes from increasing to decreasing where the function has a critical value
a) $f'=0$
b) f' undef.

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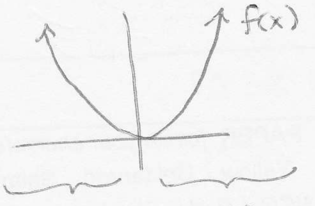
X Ex.④ $f(x) = x^2$

By examining the graph



x<0 — f decreases

x>0 — f increases

technically $(-\infty, 0]$

$$\boxed{f \begin{array}{ll} \text{decreasing} & (-\infty, 0) \\ \text{increasing} & (0, \infty) \end{array}}$$

technically $[0, \infty)$

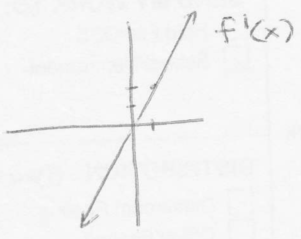
↑

The point $x=0$ is important in determining the intervals where f increases and decreases.

$x=0$ is a critical value.

By examining the graph of $f'(x) = 2x$

line
$m=2$
$b=0$.



x<0	x>0
$f'(x)<0$	$f'(x)>0$
y coord on graph is negative	y coord on graph is positive
⇒	⇒
graph below x-axis	graph above x-axis

technically $(-\infty, 0]$

$$\boxed{f \begin{array}{ll} \text{decreasing} & (-\infty, 0) \\ \text{increasing} & (0, \infty) \end{array}} \begin{array}{l} \leftarrow f'<0 \\ \leftarrow f'>0 \end{array}$$

technically $[0, \infty)$

Summary: Intervals of increasing or decreasing have critical values (and $\pm\infty$) as endpoints

Intervals where $f'(x)<0$ mean $f(x)$ decreasing
{ slope of tangent line negative — downhill }

Intervals where $f'(x)>0$ mean $f(x)$ increasing
{ slope of tangent line positive — uphill }

[redacted]

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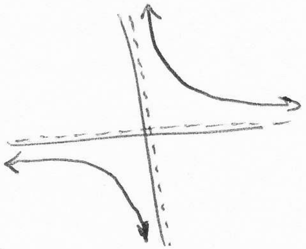
Use the graph and calculus to determine
a) intervals of increase [illegible] decrease
b) relative extrema. ← "or" means do both.

⑤ $f(x) = \frac{1}{x}$

Examine the graph



decreasing $(-\infty, 0)$
decreasing $(0, \infty)$
not defined at $x=0$, so open interval.
no relative extrema.

Calculus

$f(x) = x^{-1}$

$f'(x) = -x^{-2} = \frac{-1}{x^2}$

$f'(x) = 0$

$-\frac{1}{x^2} = 0$

$-1 = 0 \cdot x^2$

$-1 \neq 0$ no solution $\longrightarrow$ no stationary points where $f'(x) = 0$.

$f'(x)$ undefined denom $= 0$
$x^2 = 0 \longrightarrow x = 0$.

$\boxed{\text{One critical value at } x = 0}$

Sign chart

$$f' \quad \xleftarrow{\quad (-) \quad} \underset{0}{|} \xrightarrow{\quad (-) \quad}$$

no sign change $\longrightarrow$ $x=0$ is not a rel max or min.

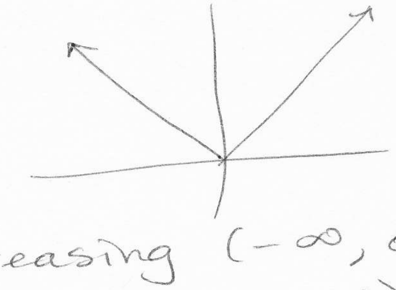
$\boxed{\text{no relative extrema}}$

$\boxed{\text{decreasing } (-\infty, 0) \cup (0, \infty)}$ some books take $(-\infty, \infty)$? Does Briggs?

M250 same instructions

⑥ $f(x) = |x|$.

Graph:



decreasing $(-\infty, 0]$
increasing $[0, \infty)$
relative minimum at $x=0$, min value $= 0$.

Calculus:

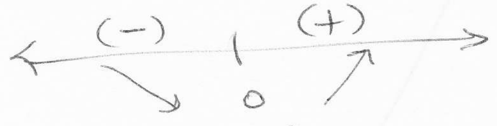
$$f(x) = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

$f'(x) \neq 0$ anywhere
$f'(x)$ undefined at $x=0$ because left $\neq$ right

critical value at $x=0$

f'



change of sign at $x=0$ $\Rightarrow$ min
decreasing $\rightarrow$ increasing

relative minimum at $x=0$
value $f(0) = |0| = 0$
decreasing $(-\infty, 0]$
increasing $[0, \infty)$

<u>at</u> $x=0$? That's the reason most books just use open intervals

✓ ⑦ Find the intervals where $f(x)=(x-1)^3(x+2)^2$ is increasing or decreasing. [redacted]

step 1: Find critical values

$f'(x)=(x-1)^3\cdot 2(x+2)\ +\ (x+2)^2\cdot 3(x-1)^2$

$f'(x)=(x-1)^2(x+2)\left[2(x-1)+3(x+2)\right]$

$f'(x)=(x-1)^2(x+2)\left[2x-2+3x+6\right]$

$f'(x)=(x-1)^2(x+2)\left[5x+4\right]$

$f'(x)$ is defined everywhere

$f'(x)=0$ at $x=1,-2,-\frac{4}{5}$.

product rule

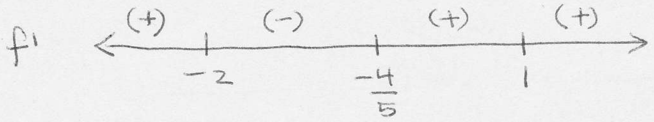
factor out least powers

dist

combine

step 2:

step 3:



Plot c.v.s in numerical order.

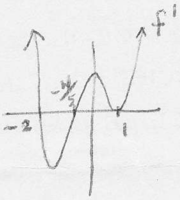
Determine sign of derivative by test point on each interval or by graph of $f'(x)$.

$f'(-3)=(-)^2(-)(-)=(+)$ above x-axis

$f'(-1)=(-)^2(+)(-)=(-)$ below

$f'(0)=(-)^2(+)(+)=(+)$ above

$f'(2)=(+)^2(+)(+)=(+)$ above



step 4:

f is increasing $(-\infty,-2)\cup(-\frac{4}{5},1)\cup(1,\infty)$

f is decreasing $(-2,-\frac{4}{5})$

Write open intervals.

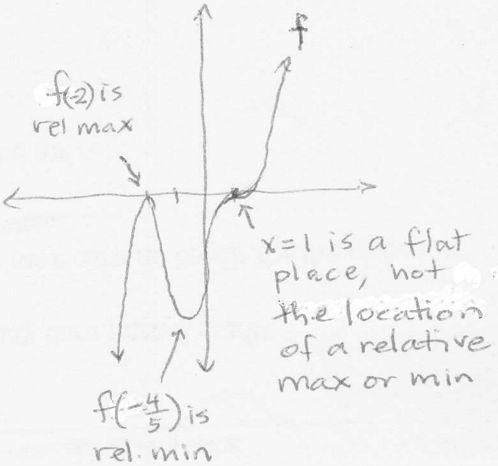
$f'(+)\rightarrow$ inc.

$f'(-)\rightarrow$ dec.

Note: $x=1$ is a critical value, but this is not enough to make $f(1)$ an extremum.

In contrast $f(-2)$ is a rel. max because f' changes from (+) to (−) increasing to decreasing as it goes "over the hump".

Similarly $f(-\frac{4}{5})$ is a rel. min because f' changes from (−) to (+) decreasing to increasing



$f(-2)$ is rel max

$x=1$ is a flat place, not the location of a relative max or min

$f(-\frac{4}{5})$ is rel. min

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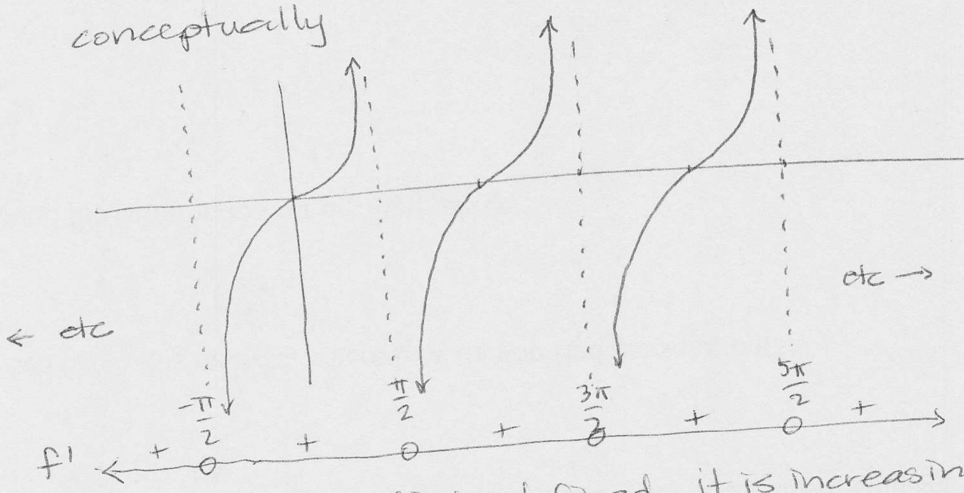
⑧ Find the intervals where $f(x) = \tan x$ is increasing and decreasing.

conceptually



whenever tan(x) is defined, it is increasing.

f continuous on $\left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$ for any integer k.

f differentiable $f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$ on $\left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$ $\forall k \in \mathbb{Z}$
↑
"k is an integer"

$f'(x) > 0$: $f'(x) = \sec^2(x) > 0$ $\forall x$ where it is defined.

So $f(x) = \tan(x)$ is increasing on $\left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$
and decreasing nowhere.

⑨ Similarly $f(x) = \cot(x)$ is decreasing on $(k\pi, (k+1)\pi)$ $\forall k \in \mathbb{Z}$
and increasing nowhere